Matrices

2001

Let \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix} \)

Show that \( AB = kI \) for some constant \( k \), where \( I \) is the identity matrix.

Hence obtain (i) the inverse matrix \( A^{-1} \) and (ii) the matrix \( A^2B \). (4 marks)

2002

1. A matrix \( A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \). Prove by induction that \( A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix} \), where \( n \) is a positive integer. (6 marks)

2. Write down the \( 2 \times 2 \) matrix \( A \) representing a reflection in the x-axis and the \( 2 \times 2 \) matrix \( B \) representing an anti-clockwise rotation of \( 30^\circ \) about the origin.

Hence show that the image of a point \((x, y)\) under the transformation \( A \) followed by the transformation \( B \) is \( \left(\frac{kx+y}{2}, \frac{x-ky}{2}\right) \), stating the value of \( k \). (4 marks)

2003

The matrix \( A \) is such that \( A^2 = 4A - 3I \) where \( I \) is the corresponding identity matrix.

Find integers \( p \) and \( q \) such that \( A^4 = pA + qI \). (4 marks)

2004

Write down the \( 2 \times 2 \) matrix \( M_1 \) associated with an anti-clockwise rotation of \( \frac{\pi}{2} \) radians about the origin.

Write down the matrix \( M_2 \) associated with reflection in the x-axis.

Evaluate \( M_2M_1 \) and describe geometrically the effect of the transformation represented by \( M_2M_1 \). (2, 1, 2 marks)

2005

Given the matrix \( A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \), show that \( A^2 + A = kI \) for some constant \( k \), where \( I \) is the \( 3 \times 3 \) unit matrix.

Obtain the values of \( p \) and \( q \) for which \( A^{-1} = pA + qI \). (4, 2 marks)

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2006

1. Calculate the inverse of the matrix \( \begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix} \).
   For what value of \( x \) is this matrix singular? (4 marks)

2. The square matrices \( A \) and \( B \) are such that \( AB = BA \). Prove by induction that \( A^n B = B A^n \) for all integers \( n \geq 1 \). (5 marks)

2007

Matrices \( A \) and \( B \) are defined by
\[
A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}.
\]

(a) Find the product \( AB \).
(b) Obtain the determinants of \( A \) and \( AB \).
   Hence or otherwise, obtain an expression for \( \det B \). (2, 2, 1 marks)

2008

Let the matrix \( A = \begin{pmatrix} 1 & x \\ x & 4 \end{pmatrix} \).
(a) Obtain the value(s) of \( x \) for which \( A \) is singular.
(b) When \( x = 2 \), show that \( A^2 = pA \) for some constant \( p \).
   Determine the value of \( q \) such that \( A^4 = qA \). (2, 3 marks)

2009

Given the matrix \( A = \begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix} \).
(a) Find \( A^{-1} \) in terms of \( t \) when \( A \) is non-singular.
(b) Write down the value of \( t \) such that \( A \) is singular.
(c) Given that the transpose of \( A \) is \( \begin{pmatrix} 6 & 3 \\ 6 & 5 \end{pmatrix} \), find \( t \). (3, 1, 1 marks)
2010

1. Obtain the $2 \times 2$ matrix $M$ associated with an enlargement, scale factor 2, followed by a clockwise rotation of $60^\circ$ about the origin. (4 marks)

2. Use Gaussian elimination to show that the set of equations
   
   \[
   \begin{align*}
   x - y + z &= 1 \\
   x + y + 2z &= 0 \\
   2x - y + az &= 2
   \end{align*}
   \]

   has a unique solution when $a \neq 2.5$. Explain what happens when $a = 2.5$.
   Obtain the solution when $a = 3$.

   Given $A = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, calculate $AB$.

   Hence or otherwise state the relationship between $A$ and the matrix $C = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$. (5, 1, 1, 2 marks)

2011

(a) For what value of $\lambda$ is
   
   \[
   \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{pmatrix}
   \]

   singular?

(b) For $A = \begin{pmatrix} 2 & 2\alpha + \beta & -1 \\ 3\alpha + 2\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$ obtain values of $\alpha$ and $\beta$ such that $A' = \begin{pmatrix} 2 & -5 & -1 \\ -1 & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$. (3, 3 marks)

2012

A non-singular $n \times n$ matrix $A$ satisfies the equation $A + A^{-1} = I$ where $I$ is the $n \times n$ identity matrix. Show that $A^3 = kI$ and state the value of $k$. (4 marks)
Matrices A and B are defined by \[ A = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix} \]

(a) Find \( A^2 \).

(b) Find the value of \( p \) for which \( A^2 \) is singular.

(c) Find the values of \( p \) and \( x \) if \( B = 3A' \). \((1, 2, 2 \text{ marks})\)

Given \( A \) is the matrix \[ A = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}, \]

Prove by induction that \( A^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}, \quad n \geq 1. \) \((4 \text{ marks})\)

Obtain the value(s) of \( p \) for which the matrix \( A = \begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix} \) is singular. \((4 \text{ marks})\)

Write down the \( 2 \times 2 \) matrix, \( M_1 \), associated with a reflection in the \( y \)-axis.

Write down a second \( 2 \times 2 \) matrix, \( M_2 \), associated with an anticlockwise rotation through an angle of \( \frac{\pi}{2} \) radians about the origin.

Find the \( 2 \times 2 \) matrix, \( M_3 \), associated with an anticlockwise rotation through \( \frac{\pi}{2} \) radians about the origin followed by a reflection in the \( y \)-axis.

What single transformation is associated with \( M_3 \)? \((4 \text{ marks})\)