### AH Maths Formulae

#### DIFFERENTIATION

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>( f'(x) )</td>
</tr>
<tr>
<td>( x^n )</td>
<td>( nx^{n-1} )</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( e^x )</td>
</tr>
<tr>
<td>( e^{ax + b} )</td>
<td>( ae^{ax + b} )</td>
</tr>
<tr>
<td>( \ln x )</td>
<td>( \frac{1}{x} )</td>
</tr>
<tr>
<td>( \ln (ax + b) )</td>
<td>( \frac{a}{ax + b} )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( -\sin x )</td>
</tr>
<tr>
<td>( \tan x )</td>
<td>( \sec^2 x )</td>
</tr>
<tr>
<td>( \sec x ) *</td>
<td>( \sec x \tan x )</td>
</tr>
<tr>
<td>( \cosec x ) *</td>
<td>( -\cosec x \cot x )</td>
</tr>
<tr>
<td>( \cot x ) *</td>
<td>( -\cosec^2 x )</td>
</tr>
<tr>
<td>( \sin^{-1} x )</td>
<td>( \frac{1}{\sqrt{1 - x^2}} )</td>
</tr>
<tr>
<td>( \sin^{-1} \frac{x}{a} )</td>
<td>( \frac{1}{\sqrt{a^2 - x^2}} )</td>
</tr>
<tr>
<td>( \cos^{-1} x )</td>
<td>( -\frac{1}{\sqrt{1 - x^2}} )</td>
</tr>
<tr>
<td>( \cos^{-1} \frac{x}{a} )</td>
<td>( -\frac{1}{\sqrt{a^2 - x^2}} )</td>
</tr>
<tr>
<td>( \tan^{-1} x )</td>
<td>( \frac{1}{1 + x^2} )</td>
</tr>
<tr>
<td>( \tan^{-1} \frac{x}{a} )</td>
<td>( \frac{a}{a^2 + x^2} )</td>
</tr>
</tbody>
</table>

* Alternatively these can be worked out

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www.national5maths.co.uk for Free Maths Resources all Levels
First principles

\[ f'(x) = \lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} \right) \]

\[ \lim_{h \to 0} \left( \frac{\sin 2h}{h} \right) = 2 \]

\[ \lim_{h \to 0} \left( \frac{\cos (3h - 1)}{h} \right) = 0 \]

Chain Rule

\[ \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \]

Product Rule

\[ \frac{d}{dx} (f(x)g(x)) = f''(x)g(x) + f(x)g'(x) \]

Quotient Rule

\[ \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \]

Parametric Differentiation

\[ \frac{dy}{dx} = \left( \frac{dy}{dt} \right) + \left( \frac{dx}{dt} \right) \]

\[ f''(x) = \frac{y''(t)x'(t) - y'(t)x''(t)}{(x'(t))^3} \]

*An easier method is to use

\[ \frac{d^2 y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \times \frac{dt}{dx} \]

Implicit Differentiation

\[ \frac{d}{dx} (f(y)) = \frac{d}{dy} (f(y)) \frac{dy}{dx} \]
Derivative of Inverse \[
\frac{d}{dx} (f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(y)}
\]

\[
\frac{dy}{dx} = 1 + \left( \frac{dx}{dy} \right)
\]

Logarithmic Differentiation
Use Log rules & Implicit Differentiation

Stationary Points
\[f''(x) > 0 \Rightarrow \text{Min } T P\]
\[f''(x) < 0 \Rightarrow \text{Max } T P\]
\[f''(x) = 0 \text{ Use Nature Table}\]

Rectilinear Motion
\[v = \frac{ds}{dt}\]
\[a = \frac{dv}{dt}\]

Magnitude of velocity = Speed = \[\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\]

Direction of Motion = \[\frac{dy}{dx}\]

Magnitude of acceleration = \[\sqrt{\left(\frac{d^2 x}{dt^2}\right)^2 + \left(\frac{d^2 y}{dt^2}\right)^2}\]
# INTEGRATION

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$\int f(x) , dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{x^{n+1}}{n+1} + C \quad n \neq -1$</td>
</tr>
<tr>
<td>$(ax + b)^n$</td>
<td>$\frac{(ax + b)^{n+1}}{a(n+1)} + C$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x + C$</td>
</tr>
<tr>
<td>$e^{ax+b}$</td>
<td>$\frac{1}{a} e^{ax+b} + C$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln x + C$</td>
</tr>
<tr>
<td>$\frac{1}{ax+b}$</td>
<td>$\frac{1}{a} \ln ax + b + C$</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>$\int f(x) , dx$ or by Substitution*</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$- \cos x + C$</td>
</tr>
<tr>
<td>$\sin(ax+b)$</td>
<td>$- \frac{1}{a} \cos(ax+b) + C$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$\sin x + C$</td>
</tr>
<tr>
<td>$\cos(ax+b)$</td>
<td>$\frac{1}{a} \sin(ax+b) + C$</td>
</tr>
<tr>
<td>$\tan x$</td>
<td>$\ln \sec x + C$</td>
</tr>
<tr>
<td>$\sec^2 x$</td>
<td>$\tan x + C$</td>
</tr>
<tr>
<td>$\sec^2(ax+b)$</td>
<td>$\frac{1}{a} \tan(ax+b) + C$</td>
</tr>
<tr>
<td>$\tan^2 x$</td>
<td>$\tan x - x + C$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
<td>$\sin^{-1} x + C$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{a^2-x^2}}$</td>
<td>$\sin^{-1} \frac{x}{a} + C$ Note $1 \cdot x^2$</td>
</tr>
<tr>
<td>$\frac{1}{1+x^2}$</td>
<td>$\tan^{-1} x + C$</td>
</tr>
<tr>
<td>$\frac{1}{a^2+x^2}$</td>
<td>$\frac{1}{a} \tan^{-1} \frac{x}{a} + C$</td>
</tr>
</tbody>
</table>
Volume of Revolution

About x-axis

\[ V = \int_a^b \pi y^2 \, dx \text{ where } y = f(x) \]

About y-axis

\[ V = \int_a^b \pi x^2 \, dy \text{ where } x = f(y) \]

Integration by parts

\[ \int f(x) g'(x) \, dx = f(x) g(x) - \int f'(x) g(x) \, dx \]

\[ \int_a^b f(x) g'(x) \, dx = [f(x) g(x)] - \int_a^b f'(x) g(x) \, dx \]

By substitution

\[ dx \to \frac{dx}{du} \]

For definite integrals remember to change limits.

<table>
<thead>
<tr>
<th>[ \int , dx ]</th>
<th>[ \text{Substitution} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ f(ax + b) ]</td>
<td>[ u = ax + b ]</td>
</tr>
<tr>
<td>[ f'(x) , [f(x)]^n ]</td>
<td>[ u = f(x) ]</td>
</tr>
<tr>
<td>[ (ax + b) , (cx + d)^n ]</td>
<td>[ u = (cx + d) ]</td>
</tr>
<tr>
<td>[ f'(x) , e^{f(x)} ]</td>
<td>[ u = f(x) ]</td>
</tr>
</tbody>
</table>
| \[ \frac{f'(x)}{f(x)} \] | \[ u = f(x) \] *

Rectilinear Motion

\[ v = \int a \, dt \]

\[ s = \int v \, dt \]
**DIFFERENTIAL EQUATIONS**

First Order Separable

\[ \frac{dy}{dx} = f(x) \, g(y) \quad \Rightarrow \quad \int \frac{dy}{g(y)} = \int f(x) \, dx \]

Exponential Growth

\[ \frac{dP}{dt} = kP \quad \text{Soln:} \quad P = Ae^{kt} \]

Exponential Decay

\[ \frac{dP}{dt} = -kP \quad \text{Soln:} \quad P = Ae^{-kt} \]

Newton’s Law of Cooling

\[ \frac{dT}{dt} = -k(T - T_0) \]

First Order Integrating Factor

\[ \frac{dy}{dx} + P(x) \, y = f(x) \quad \Rightarrow \quad I(x) \, y = \int I(x) \, f(x) \, dx \]

where \( I(x) = \exp \int P(x) \, dx \)

Second Order Homogeneous

\[ a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0 \]

\[ \Rightarrow \quad am^2 + bm + c = 0 \quad (\text{Auxiliary Equation}) \]

Real, distinct roots \( m_1, m_2 \)

\[ \Rightarrow \quad y = A \exp (m_1 \, x) + B \exp (m_2 \, x) \]

Real, equal roots \( m \)

\[ \Rightarrow \quad y = A \exp (m \, x) + B \, x \exp (m \, x) \]

Complex roots \( m = p + iq \)

\[ y = \exp(px) \, (A \cos qx + B \sin qx) \]
Second Order Non-Homogeneous \[ a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \]

\[ \Rightarrow y = y_e + y_p \]

where \( y_e \) is the \textbf{Complementary Function} (Solution to homogeneous case)

and \( y_p \) is the \textbf{Particular Solution}.

\[ f(x) = \text{polynomial of degree } n \implies y_p = C x^n + D x^{n-1} + \ldots + T x + S \]

\[ f(x) = e^{rx} \implies y_p = D e^{rx} \]

\[ f(x) = l \sin nx + m \cos nx \implies y_p = p \sin nx + q \cos nx \]

\textbf{Note} If the Particular Integral we try contains terms already in the Complementary Function we need to introduce a factor of \( x \) or \( x^2 \) to our P.I.

\[ \text{TRIGONOMETRY} \]

\[ \sin^2 x + \cos^2 x = 1 \]

\[ \tan x = \frac{\sin x}{\cos x} \]

\[ \sin 2x = 2 \sin x \cos x \]

\[ \cos 2x = \cos^2 x - \sin^2 x \]

\[ = 2 \cos^2 x - 1 \]

\[ = 1 - 2 \sin^2 x \]

\[ \cosec x = \frac{1}{\sin x} \quad \sec x = \frac{1}{\cos x} \quad \cot x = \frac{1}{\tan x} \]

\[ \sec^2 x = \tan^2 x + 1 \]

\[ \tan^2 x = \sec^2 x - 1 \]
PARTIAL FRACTIONS

<table>
<thead>
<tr>
<th>Number of Factors in Denominator</th>
<th>Type of Factors in Denominator</th>
<th>Partial Fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2 distinct, linear</td>
<td>( \frac{A}{(_)} + \frac{B}{(_)} )</td>
</tr>
<tr>
<td>2</td>
<td>1 repeated, linear</td>
<td>( \frac{A}{(_)} + \frac{B}{(_)^2} )</td>
</tr>
<tr>
<td>3</td>
<td>3 distinct, linear</td>
<td>( \frac{A}{(_)} + \frac{B}{(_)} + \frac{C}{(_)} )</td>
</tr>
<tr>
<td>3</td>
<td>1 distinct, 1 repeated</td>
<td>( \frac{A}{(_)} + \frac{B}{(_)^2} + \frac{C}{(_)^3} )</td>
</tr>
<tr>
<td>3</td>
<td>1 repeated</td>
<td>( \frac{A}{(_)} + \frac{Bx + C}{(_)^3} )</td>
</tr>
<tr>
<td>3</td>
<td>1 linear, 1 irreducible quadratic</td>
<td>( \frac{A}{(_)} + \frac{Bx + C}{(_)^3} )</td>
</tr>
</tbody>
</table>

Improper Rational Functions

Degree \( m \)  \( m \) > \( n \)  Use Long Division

SEQUENCES AND SERIES

Arithmetic

\( u_n = a + (n - 1) d \), \( d = u_{n+1} - u_n \)

\( S_n = \frac{n}{2} [2a + (n - 1) d] \)

Geometric

\( u_n = a r^{n-1} \), \( r = \frac{u_{n+1}}{u_n} \)

\( S_n = \frac{a(1 - r^n)}{1 - r} \)

\( S_\infty = \frac{a}{1 - r} \)  \( -1 < r < 1 \), \( r \neq 0 \)
MacLaurin’s Theorem

\[ f(x) = \sum_{r=0}^{\infty} f^{(r)}(0) \frac{x^r}{r!} = f(0) + f^{1}(0) \frac{x}{1!} + f^{2}(0) \frac{x^2}{2!} + f^{3}(0) \frac{x^3}{3!} + \ldots + f^{n}(0) \frac{x^n}{n!} \]

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \]

\[ \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \]

\[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \]

**NUMBER THEORY**

\[ \sum_{r=1}^{n} r = \frac{1}{2} n(n + 1) \]

\[ \sum_{r=1}^{n} r^2 = \frac{1}{6} n(n + 1)(2n + 1) \]

\[ \sum_{r=1}^{n} r^3 = \frac{1}{4} n^2(n + 1)^2 \]
Binomial Expansion

Pascal’s Triangle

\[
\begin{array}{cccc}
1 & & & \\
1 & 1 & & \\
1 & 2 & 1 & \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
& & & & etc
\end{array}
\]

\[(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r\]

\[\binom{n}{r} = \frac{n!}{r!(n-r)!}\]

Natural Numbers = \(N\)
\[
\{1, 2, 3, 4, 5, 6, \ldots\}
\]

Whole Numbers = \(W\)
\[
\{0, 1, 2, 3, 4, 5, 6, \ldots\}
\]

Integers = \(Z\)
\[
\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}
\]

Rational Numbers = \(Q\)
\[
\{\text{Any numbers that can be written as } a/b\}
\]

Real Numbers = \(R\)
\[
\{\text{All rational and irrational numbers}\}
\]

FUNCTIONS

Even Functions \[f(-x) = f(x)\] Symmetrical about \(x - axis\)

Odd Functions \[f(-x) = -f(x)\] Half-turn symmetry about \(O\)

Inverse Functions \[f^{-1} f(x) = x\]

CURVE SKETCHING

1. Vertical asymptote \(\text{Denominator} = 0\)
2. Non-vertical asymptote \(\text{Division}\)
3. Turning Points \(f'(x) = 0\)
4. Axes intercepts \(f(0)\) and \(f(x) = 0\)
5. Sketch
SYSTEEMS OF EQUATIONS

Gaussian Elimination

Reduce matrix to

\[
\begin{bmatrix}
a & b & c \\
0 & d & e \\
0 & 0 & e
\end{bmatrix}
\]

Solutions

\[x = \frac{b}{a}\]

\[a = 0, \ b \neq 0\] No solution

\[a = 0, \ b = 0\] Multiple Solutions

\[a \neq 0, \ b \neq 0\] Unique Solutions

\[a \neq 0, \ b = 0\] Solution = 0

COMPLEX NUMBERS

\[i^2 = -1\]

\[i = \sqrt{-1}\]

\[z = x + iy\]

Complex Conjugate

\[\bar{z} = x - iy\]

Polar Form

\[z = r (\cos \theta + i \sin \theta)\]

\[r = \sqrt{x^2 + y^2}\]

\[\tan \theta = \frac{y}{x}\]

De Moivre’s Theorem

\[z^n = r^n (\cos n\theta + i \sin n\theta)\]

Graphical Representations

\[z = r\] Circle, centre (0,0) and radius r

\[z - a = r\] Circle, centre a and radius r

\[z - a > r\] All points outside circle centre a, radius r

\[z - a < r\] All points inside circle centre a, radius r

\[z - a = z - b\] Set of points that lie on the perpendicular bisector of line joining a and b
MATRICES

Zero Matrix
\[ O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \text{Identity Matrix for Addition} \]

Identity Matrix for Multiplication
\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Order
\[ m \times n \quad \text{m rows} \quad \text{n columns} \]

Transpose
\[ A^T \quad \text{or} \quad A' \quad \text{Rows and Columns are interchanged} \]

Symmetric
\[ A = A^T \]

Skew-symmetric
\[ A^T = -A \]

PROPERTIES
\[ A + B = B + A \]
\[ A - B \neq B - A \]
\[ AB \neq BA \quad \text{in general} \]
\[ (A + B) + C = A + (B + C) = A + B + C \]
\[ (AB)C = A(BC) = ABC \quad \text{with condition} \]
\[ k(A + B) = Ka + kB, \quad k \text{is a scalar} \]
\[ A(B + C) = AB + AC \quad \text{with condition} \]
\[ (A^T)^T = A \]
\[ (A + B)^T = A^T + B^T \]
\[ (AB)^T = B^T A^T \]
\[ (kA)^T = kA^T \]
\[ AI = IA = A \]
\[ \det(AB) = \det A \det B \]
\[ AA^{-1} = A^{-1} A = I \]
\[ (AB)^{-1} = B^{-1} A^{-1} \]
\[ (A^{-1})^T = (A^T)^{-1} \]
DETERMINANTS

2 x 2

\[ A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \]

\[ \text{det } A = ad - bc \]

3 x 3

\[ A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & f & i \end{bmatrix} \]

\[ \text{det } A = a \text{det} \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \text{det} \begin{bmatrix} f & g \\ i & h \end{bmatrix} + c \text{det} \begin{bmatrix} d & e \\ g & h \end{bmatrix} \]

Singular

\[ \text{determinant } = 0 \]

Non - singular

\[ \text{determinant } \neq 0 \]

INVERSE

2 x 2

\[ A^{-1} = \frac{1}{\text{det } A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

3 x 3

Form Augmented Matrix \( (A \ I) \)

Perform EROs to reduce \( A \) to I

R.H.S. then is \( A^{-1} \)

Orthogonal

\[ A^T = A^{-1} \quad \text{or} \quad A^T A = I \]

TRANSFORMATIONS

Rotation

\[ \begin{bmatrix} \cos \theta & - \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \] rotates point anti-clockwise \( \theta \) about the origin.
Reflection

\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\]
reflects point in the y-axis.

\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]
reflects a point in the x-axis.

\[
\begin{bmatrix}
\cos 2\theta & \sin 2\theta \\
\sin 2\theta & -\cos 2\theta
\end{bmatrix}
\]
reflects a point in the line through the origin that makes an angle of \(\theta\) with positive x-axis.

**VECTORS**

**Components**

\[\overrightarrow{AB} = \mathbf{b} - \mathbf{a}\]

**Magnitude**

\[\overrightarrow{AB} = \sqrt{i^2 + j^2 + k^2}\]

**Scalar Product (Dot product)**

\[\mathbf{a} \cdot \mathbf{b} = a \cdot b \cos \theta\]

\[\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2\]

**Vector Product (Cross Product)**

Given \(\mathbf{a} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}\) and \(\mathbf{b} = \begin{bmatrix} b \\ b \\ b \end{bmatrix}\)

\[\mathbf{a} \times \mathbf{b} = i \quad i \quad k\]

\[a_1 \quad a_2 \quad a_3\]

\[b_1 \quad b_2 \quad b_3\]