PROPERTIES OF FUNCTIONS

SKETCHING RATIONAL FUNCTIONS

A rational function is a function of the form \( \frac{p(x)}{q(x)} \), where \( p(x) \) and \( q(x) \) are polynomials in \( x \).

Consider the graph of the simple rational function \( y = \frac{1}{x} \), \( x \neq 0 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1000</td>
<td>100</td>
<td>10</td>
<td>1</td>
<td>0.1</td>
<td>0.01</td>
<td>0.001</td>
</tr>
</tbody>
</table>

The behaviour of \( y \) for negative values of \( x \) can be investigated similarly.

The graph approaches the x-axis and y-axis but does not actually touch either axis. The x-axis and y-axis are known as asymptotes.

Note that the graph "jumps" suddenly at either side of \( x = 0 \). The graph is said to be discontinuous at \( x = 0 \).
General Notes

(1) Vertical asymptotes always occur when the denominator of the function equals zero.

(2) At either side of a vertical asymptote, $y \to \infty$ or $y \to -\infty$. The behaviour of a graph at either side of a vertical asymptote should always be investigated.

(3) Non-vertical asymptotes occur when $x \to \pm \infty$.

(4) Any points of intersection with the coordinate axes should be investigated.

(5) The coordinates and nature of any stationary points should be found when requested.
Worked Example 1

Sketch the graph of \( y = \frac{1}{(x + 2)(x - 3)} \).
[You need not find the coordinates of any stationary points.]

Solution

\( y \)-axis:
When \( x = 0 \), \( y = \frac{1}{(2)(-3)} = -\frac{1}{6} \).
The curve cuts the \( y \)-axis at \( \left( 0, -\frac{1}{6} \right) \).

\( x \)-axis:
When \( y = 0 \), \( \frac{1}{(x + 2)(x - 3)} = 0 \) \( \Rightarrow \ 1 = 0 \) ???
This means that the curve does not cut the \( x \)-axis.

Vertical Asymptotes: \( (x + 2)(x - 3) = 0 \) \( \Rightarrow \ x = -2 \) or \( x = 3 \)

The behaviour of the curve at either side of these vertical asymptotes must be investigated. We know that \( y \to \infty \) or \( y \to -\infty \) before or after a vertical asymptote. The simplest way to investigate the behaviour is to calculate \( y \) for a value of \( x \) just before a vertical asymptote and for a value of \( y \) just after the vertical asymptote. If the calculated value of \( y \) is positive, it means that \( y \to \infty \) and if the value of \( y \) is negative, it means that \( y \to -\infty \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2.1</th>
<th>-2</th>
<th>-1.9</th>
<th>2.9</th>
<th>3</th>
<th>3.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>+\infty</td>
<td>-\infty</td>
<td>-\infty</td>
<td>+\infty</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Non-Vertical Asymptote:
\[
y = \frac{1}{(x + 2)(x - 3)}
\]

As \( x \to \pm \infty \), \( y \to 0 \).
This means that \( y = 0 \) is a non-vertical asymptote.
\[ y = \frac{1}{(x+2)(x-3)} \]
**Worked Example 2**

Sketch the graph of $y = \frac{x - 3}{x^2 + x - 2}$.

[You need not find the coordinates of any stationary points.]

**Solution**

**y-axis:**

When $x = 0$, $y = \frac{-3}{-2} = \frac{3}{2}$.

The curve cuts the $y$-axis at $(0, \frac{3}{2})$.

**x-axis:**

When $y = 0$, $\frac{x - 3}{x^2 + x - 2} = 0 \Rightarrow x - 3 = 0 \Rightarrow x = 3$

The curve cuts the $x$-axis at $(3, 0)$.

**Vertical Asymptotes:**

$x^2 + x - 2 = 0 \Rightarrow (x + 2)(x - 1) = 0 \Rightarrow x = -2$ or $x = 1$

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2.1</th>
<th>-2</th>
<th>-1.9</th>
<th>0.9</th>
<th>1</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$-\infty$</td>
<td>$+\infty$</td>
<td>$+\infty$</td>
<td>$-\infty$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Non-Vertical Asymptote:**

$y = \frac{x - 3}{x^2 + x - 2}$

As $x \to \pm\infty$, $y \to 0$ (since the degree of the denominator is higher than the degree of the numerator).

This means that $y = 0$ is a non-vertical asymptote.
YOU CAN NOW ATTEMPT THE WORKSHEET "SKETCHING RATIONAL FUNCTIONS 1".

\[ y = \frac{x - 3}{x^2 + x - 2} \]
Before investigating the non-vertical asymptote of an improper rational function, algebraic long division must be used.

**Worked Example 3**

Sketch the graph of \( y = \frac{x + 4}{x + 2} \).
[You need not find the coordinates of any stationary points.]

**Solution**

**y-axis:**

When \( x = 0 \), \( y = \frac{4}{2} = 2 \).
The curve cuts the y-axis at (0, 2).

**x-axis:**

When \( y = 0 \), \( \frac{x + 4}{x + 2} = 0 \) \( \Rightarrow \) \( x + 4 = 0 \) \( \Rightarrow \) \( x = -4 \)

The curve cuts the x-axis at (−4, 0).

**Vertical Asymptotes:** \( x + 2 = 0 \) \( \Rightarrow \) \( x = -2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2.1</th>
<th>-2</th>
<th>-1.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>−∞</td>
<td>+∞</td>
<td></td>
</tr>
</tbody>
</table>

**Non-Vertical Asymptote:**

\[
y = \frac{x + 4}{x + 2}
\]

\[
\begin{align*}
1 \\
\frac{x + 2}{x + 4} \\
\frac{x + 2}{x + 2} \\
2
\end{align*}
\]

\[
y = 1 + \frac{2}{x + 2}
\]

As \( x \rightarrow \pm\infty \), \( y \rightarrow 1 \).
This means that \( y = 1 \) is a non-vertical asymptote.
\[ y = \frac{x + 4}{x + 2} \]
Worked Example 4

Sketch the graph of \( y = \frac{x^2 - 4}{x - 1} \).
[You need not find the coordinates of any stationary points.]

Solution

\( y \)-axis: When \( x = 0 \), \( y = \frac{-4}{-1} = 4 \).
The curve cuts the \( y \)-axis at \( (0, 4) \).

\( x \)-axis: When \( y = 0 \), \( \frac{x^2 - 4}{x - 1} = 0 \) \( \Rightarrow \) \( x^2 - 4 = 0 \)
\( \Rightarrow \) \( x^2 = 4 \)
\( \Rightarrow \) \( x = \pm 2 \)
The curve cuts the \( x \)-axis at \( (-2, 0) \) and \( (2, 0) \).

Vertical Asymptotes: \( x - 1 = 0 \) \( \Rightarrow \) \( x = 1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.9</th>
<th>1</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>+∞</td>
<td>−∞</td>
<td></td>
</tr>
</tbody>
</table>

Non-Vertical Asymptote:

\[
y = \frac{x^2 - 4}{x - 1}
\]

\[
x - 1 \overline{\underline{x^2 + 0x - 4}}
\begin{array}{c}
x - 4 \\
\underline{x - 1}
\end{array}
\]

\[
y = x + 1 - \frac{3}{x - 1}
\]

As \( x \rightarrow \pm \infty \), \( y \rightarrow x + 1 \).
This means that \( y = x + 1 \) is a non-vertical asymptote.
YOU CAN NOW ATTEMPT THE WORKSHEET "SKETCHING RATIONAL FUNCTIONS 2".
Worked Example 5

A function \( f \) is defined by \( f(x) = \frac{2x^2 + x - 1}{x - 1} \).

(a) Find the coordinates of all the points where the graph of \( y = f(x) \) crosses the coordinate axes.

(b) Find the equation of each asymptote.

(c) Find the coordinates of each of the stationary points on the graph of \( y = f(x) \) and determine their nature.

(d) Sketch the graph of \( y = f(x) \).

(e) State the range of values of the constant \( k \) such that the equation \( f(x) = k \) has no real solutions for \( x \).

Solution

(a) \( y \)-axis: \( \text{When } x = 0, \ y = \frac{-1}{-1} = 1. \)

The curve crosses the \( y \)-axis at \( (0, 1) \).

\( x \)-axis: \( \text{When } y = 0, \ \frac{2x^2 + x - 1}{x - 1} = 0 \)

\[ \Rightarrow \quad 2x^2 + x - 1 = 0 \]

\[ \Rightarrow \quad (2x - 1)(x + 1) = 0 \]

\[ \Rightarrow \quad x = \frac{1}{2} \text{ or } x = -1 \]

The curve crosses the \( x \)-axis at \( (-1, 0) \) and \( \left( \frac{1}{2}, 0 \right) \).

(b) \( \text{Vertical Asymptotes: } x - 1 = 0 \quad \Rightarrow \quad x = 1 \)

\[
\begin{array}{c|ccc}
  x & 0.9 & 1 & 1.1 \\
  y & +\infty & +\infty \\
\end{array}
\]
Non-Vertical Asymptote:

\[
y = \frac{2x^2 + x - 1}{x - 1}
\]

\[
\begin{array}{c}
2x + 3 \\
-1 \\
\hline
2x^2 + x - 1 \\
-2x^2 - 2x \\
\hline
3x - 1 \\
3x - 3 \\
\hline
2
\end{array}
\]

\[
y = 2x + 3 + \frac{2}{x - 1}
\]

As \( x \to \pm \infty \), \( y \to 2x + 3 \).
This means that \( y = 2x + 3 \) is a non-vertical asymptote.

(c) There are two methods of finding the coordinates and nature of the stationary points.

Method 1:

The form \( y = \frac{2x^2 + x - 1}{x - 1} \) can be differentiated using the quotient rule.

\[
\frac{dy}{dx} = \frac{(x - 1)(4x + 1) - (2x^2 + x - 1) \cdot 1}{(x - 1)^2}
\]

\[
= \frac{4x^2 - 3x - 1 - 2x^2 - x + 1}{(x - 1)^2}
\]

\[
= \frac{2x^2 - 4x}{(x - 1)^2}
\]

\[
= \frac{2x(x - 2)}{(x - 1)^2}
\]

At a stationary point, \( \frac{dy}{dx} = 0 \) \( \Rightarrow \)

\[
\frac{2x(x - 2)}{(x - 1)^2} = 0
\]

\( \Rightarrow \)

\[
2x(x - 2) = 0
\]

\( \Rightarrow \)

\[
x = 0 \text{ or } x = 2
\]

When \( x = 0 \), \( y = 1 \to (0, 1) \).
When \( x = 2 \), \( y = 9 \to (2, 9) \).
The nature of each stationary point can be determined using a nature table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-0.1$</th>
<th>$0$</th>
<th>$0.1$</th>
<th>$1.9$</th>
<th>$2$</th>
<th>$2.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dy}{dx}$</td>
<td>$+$</td>
<td>$0$</td>
<td>$-$</td>
<td>$-$</td>
<td>$0$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

$(0, 1)$ is a maximum turning point and $(2, 9)$ is a minimum turning point.

**Method 2:**

The form $y = 2x + 3 + \frac{2}{x - 1}$ can be differentiated using the chain rule.

$$y = 2x + 3 + 2(x - 1)^{-1} \quad \Rightarrow \quad \frac{dy}{dx} = 2 - 2(x - 1)^{-2} \cdot 1$$

$$= 2 - \frac{2}{(x - 1)^2}$$

At a stationary point, $\frac{dy}{dx} = 0 \quad \Rightarrow \quad 2 - \frac{2}{(x - 1)^2} = 0 \quad [\times (x - 1)^2]$  

$$\Rightarrow \quad 2(x - 1)^2 - 2 = 0$$  

$$\Rightarrow \quad 2(x - 1)^2 = 2$$  

$$\Rightarrow \quad (x - 1)^2 = 1$$  

$$\Rightarrow \quad x = 2 \text{ or } x = 0$$

When $x = 0$, $y = 1 \rightarrow (0, 1)$.  
When $x = 2$, $y = 9 \rightarrow (2, 9)$.

The nature of each stationary point can be found using the second derivative.

$$\frac{dy}{dx} = 2 - 2(x - 1)^{-2} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 4(x - 1)^{-3} \cdot 1 = \frac{4}{(x - 1)^3}$$

When $x = 0$:  
$$\frac{d^2y}{dx^2} = \frac{4}{(-1)^3} = -4 < 0 \quad \Rightarrow \quad (0, 1) \text{ is a maximum t.p.}$$

When $x = 2$:  
$$\frac{d^2y}{dx^2} = \frac{4}{1^3} = 4 > 0 \quad \Rightarrow \quad (2, 9) \text{ is a minimum t.p.}$$
(e) The graph shows that the equation $f(x) = k$ has no real solutions when $k$ lies in the interval $1 < k < 9$.

YOU CAN NOW ATTEMPT THE WORKSHEET "SKETCHING RATIONAL FUNCTIONS 3".
THE GRAPH OF $y = |f(x)|$

Recall that $|x|$ denotes the magnitude of a real number $x$ and is the positive numerical value of $x$, regardless of whether $x$ itself is positive or negative.

$|2| = 2$, $|-3| = 3$, and so on.

Clearly, $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

Given a function $f(x)$, $|f(x)|$ is always non-negative and therefore the graph of $y = |f(x)|$ will be lie entirely above or on the $x$-axis.

The graph of $y = |f(x)|$ is easily obtained from the graph of $y = f(x)$ as follows:

(1) The parts of the graph of $y = f(x)$ which lie above or on the $x$-axis will remain unchanged on the graph of $y = |f(x)|$.

(2) The parts of the graph of $y = f(x)$ which lie below the $x$-axis will must be reflected in the $x$-axis to lie above the $x$-axis on the graph of $y = |f(x)|$.

The graphs below illustrate how the graph of $y = |f(x)|$ is obtained from the graph of $y = f(x)$ for a particular function $f(x)$.
ODD AND EVEN FUNCTIONS

Given a function \( f \), if \( f(-x) = f(x) \) for all values of \( x \), \( f \) is said to be an even function. The graph of an even function will always be symmetrical about the \( y \)-axis, since \( f(-1) = f(1) \), \( f(-2) = f(2) \), etc.

The graph of an even function is shown below.

\[
\begin{align*}
&\text{y} \\
\hline
&\text{x} \\
&y = f(x)
\end{align*}
\]

If \( f(-x) = -f(x) \) for all values of \( x \), \( f \) is said to be an odd function. The graph of an odd function will always have half-turn symmetry about the origin, since \( f(-1) = -f(1) \), \( f(-2) = -f(2) \), etc.

The graph of an odd function is shown below.

\[
\begin{align*}
&\text{y} \\
\hline
&\text{x} \\
&y = f(x)
\end{align*}
\]

To determine whether a given function \( f \) is odd, even or neither, find an expression for \( f(-x) \) and compare this expression to \( f(x) \). If \( f(-x) = f(x) \), then the function \( f \) is even; if \( f(-x) = -f(x) \), then the function \( f \) is odd; otherwise, the function \( f \) is neither odd nor even.
It is useful to know the following trigonometric identities for negative angles:

\[
\begin{align*}
\sin(-x) &= -\sin x \\
\cos(-x) &= \cos x \\
\tan(-x) &= -\tan x
\end{align*}
\]

You can easily verify using a calculator that, for example, \(\sin(-30^\circ) = -\sin30^\circ\), whereas \(\cos(-30^\circ) = \cos30^\circ\).

**Worked Example 1**

Prove that the function \(f(x) = x^4 - 2x^2 + 3\) is an even function.

**Solution**

\[
\begin{align*}
f(-x) &= (-x)^4 - 2(-x)^2 + 3 \\
&= x^4 - 2x^2 + 3 \quad [\text{since } (-x)^4 = x^4 \text{ and } (-x)^2 = x^2] \\
&= f(x)
\end{align*}
\]

Hence \(f(-x) = f(x)\) for all values of \(x\) and \(f\) is an even function.

**Worked Example 2**

Prove that the function \(f(x) = x^3 - 2x\) is an odd function.

**Solution**

\[
\begin{align*}
f(-x) &= (-x)^3 - 2(-x) \\
&= -x^3 + 2x \quad [\text{since } (-x)^3 = -x^3] \\
&= -(x^3 - 2x) \\
&= -f(x)
\end{align*}
\]

Hence \(f(-x) = -f(x)\) for all values of \(x\) and \(f\) is an odd function.
**Worked Example 3**

Investigate whether the function $f(x) = x^3 \sin x$ is odd, even or neither.

**Solution**

$$f(-x) = (-x)^3 \sin(-x)$$
$$= -x^3 \cdot (-\sin x) \quad \text{[since } (-x)^3 = -x^3 \text{ and } \sin(-x) = -\sin x \text{ ]}$$
$$= x^3 \sin x$$
$$= f(x)$$

Hence $f(-x) = f(x)$ for all values of $x$ and $f$ is an even function.