X100/701

NATIONAL QUALIFICATIONS 2005
FRIDAY, 20 MAY  
1.00 PM – 4.00 PM
MATHEMATICS ADVANCED HIGHER

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.
3. Full credit will be given only where the solution contains appropriate working.
Answer all the questions.

1. (a) Given \( f(x) = x^3 \tan 2x \), where \( 0 < x < \frac{\pi}{4} \), obtain \( f'(x) \).
(b) For \( y = \frac{1 + x^2}{1 + x} \), where \( x \neq -1 \), determine \( \frac{dy}{dx} \) in simplified form.

2. Given the equation \( 2y^2 - 2xy - 4y + x^2 = 0 \) of a curve, obtain the \( x \)-coordinate of each point at which the curve has a horizontal tangent.

3. Write down the Maclaurin expansion of \( e^x \) as far as the term in \( x^4 \).
Deduce the Maclaurin expansion of \( e^{x^2} \) as far as the term in \( x^4 \).
Hence, or otherwise, find the Maclaurin expansion of \( e^x + x^2 \) as far as the term in \( x^4 \).

4. The sum, \( S(n) \), of the first \( n \) terms of a sequence, \( u_1, u_2, u_3, \ldots \) is given by \( S(n) = 8n - n^2 \), \( n \geq 1 \).
Calculate the values of \( u_1, u_2, u_3 \) and state what type of sequence it is.
Obtain a formula for \( u_n \) in terms of \( n \), simplifying your answer.

5. Use the substitution \( u = 1 + x \) to evaluate \( \int_0^1 \frac{x}{\sqrt{1 + x}} \, dx \).

6. Use Gaussian elimination to solve the system of equations below when \( \lambda \neq 2 \):
\[
\begin{align*}
  x + y + 2z &= 1 \\
  2x + \lambda y + z &= 0 \\
  3x + 3y + 9z &= 5.
\end{align*}
\]
Explain what happens when \( \lambda = 2 \).

7. Given the matrix \( A = \begin{pmatrix} 0 & 1 & -1 \\ 4 & 0 & -2 \\ 2 & 1 & -3 \end{pmatrix} \), show that \( A^2 + A = kI \) for some constant \( k \), where \( I \) is the \( 3 \times 3 \) unit matrix.
Obtain the values of \( p \) and \( q \) for which \( A^{-1} = pA + qI \).

8. The equations of two planes are \( x - 4y + 2z = 1 \) and \( x + y - z = -5 \). By letting \( z = t \), or otherwise, obtain parametric equations for the line of intersection of the planes.
Show that this line lies in the plane with equation \( x + 2y - 4z = -11 \).
9. Given the equation $z + 2iz = 8 + 7i$, express $z$ in the form $a + ib$. 

10. Prove by induction that, for all positive integers $n$,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} \cdot \frac{1}{2(n+1)(n+2)}.$$ 

State the value of $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. 

11. The diagram shows part of the graph of $y = \frac{x^3}{x-2}, x \neq 2$.

(a) Write down the equation of the vertical asymptote. 

(b) Find the coordinates of the stationary points of the graph of $y = \frac{x^3}{x-2}$. 

(c) Write down the coordinates of the stationary points of the graph of $y = \left| \frac{x^3}{x-2} \right| + 1$. 

12. Let $z = \cos \theta + i \sin \theta$.

(a) Use the binomial expansion to express $z^4$ in the form $u + iv$, where $u$ and $v$ are expressions involving $\sin \theta$ and $\cos \theta$. 

(b) Use de Moivre’s theorem to write down a second expression for $z^4$. 

(c) Using the results of (a) and (b), show that

$$\frac{\cos 4\theta}{\cos^2 \theta} = p \cos^2 \theta + q \sec^2 \theta + r,$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, stating the values of $p, q$ and $r$. 

[Turn over for Questions 13, 14 and 15 on Page four]
13. Express \( \frac{1}{x^3 + x} \) in partial fractions.  

Obtain a formula for \( I(k) \), where \( I(k) = \int_{1}^{b} \frac{1}{x^3 + x} \, dx \), expressing it in the form \( \lambda \left( \frac{a}{b} \right) \) where \( a \) and \( b \) depend on \( k \).  

Write down an expression for \( e^{I(k)} \) and obtain the value of \( \lim_{k \to \infty} e^{I(k)} \).  

14. Obtain the general solution of the differential equation  
\[ \frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = 20 \sin x. \]  

Hence find the particular solution for which \( y = 0 \) and \( \frac{dy}{dx} = 0 \) when \( x = 0 \).  

15. (a) Given \( f(x) = \sqrt{\sin x} \), where \( 0 < x < \pi \), obtain \( f'(x) \).  

(b) If, in general, \( f(x) = \sqrt{g(x)} \), where \( g(x) > 0 \), show that \( f'(x) = \frac{g'(x)}{k \sqrt{g(x)}} \), stating the value of \( k \).  

Hence, or otherwise, find \( \int \frac{x}{\sqrt{1-x^2}} \, dx \).  

(c) Use integration by parts and the result of (b) to evaluate \( \int_{0}^{\pi/2} \sin^{-1} x \, dx \).  

[END OF QUESTION PAPER]